Eight Mathematical Practices

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

For the teacher:

- **Choose problems that have multiple methods of solving.**
- **Allow students time to compare/contrast different avenues to a solution.**
- **Focus on process and not just the final answer.**
- **Encourage creative floundering.**
- **Persevere!!**

For the students:

- **What are you looking for? (explain meaning/restate)**
- **What do you know? (analyze)**
- **What do you need to know? (analyze)**
- **Is this problem similar to a problem you have solved before? (find simpler forms of the original problem)**
- **What are your possible plans to find the solution? (entry points to a solution)**
- **Can you illustrate the problem? (model using diagrams)**
- **How do you find it? (plan a pathway)**
- **Is the plan working? (monitoring and evaluating progress/PERSEVERE)**
- **Does this make mathematical sense? (check your answer)**
- **Do you need to change your plan? (change course if necessary)**
- **Is this the only way to solve this problem? (collaborate with other students)**

DOES THIS MAKE MATHEMATICAL SENSE?

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

For the teacher:

- **Ability to decontextualize**
  - **Take problems and draw pictures.**
  - **Apply information in the problem to the picture.**
  - **Interpret mathematically.**
- **Ability to contextualize**
  - **Pause and allow processing time as needed while solving a problem.**
  - **Determine what the unknown variable will represent.**
  - **Focus on mathematical details, i.e. symbols, units, etc.**

For the students:

- **What does the variable(s) represent/mean? (make sense of relationships)**
  What is the meaning of the symbols i.e. =, + , ≅, ⊥, <, >, ≤, ≥, π, ~, ±, °, ≈, ≠, ∠, √, ∩, ∪, ∈, %, Δ? (represent symbols as if they have a life of their own)
- **Can you draw a picture? (quantitative reasoning)**
- **Do you need units of measure? (consider the units involved)**
- **Are the units of measure appropriate? (attend to the meaning of the quantities)**
- **Can you write an equation? (represent symbolically)**
- **Do your drawing and equation make mathematical sense? (contextualize)**
- **Does your answer make mathematical sense? (contextualize)**

DOES THIS MAKE MATHEMATICAL SENSE?

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

For the teacher:

- **Emphasize vocabulary.**
- **Paired constructive work among students.**
- **Allow time for students to analyze other students' work.**

For the students:

- **What definitions, formulas, and theorems relate to the problems?**
- **What leads to the next step? (logical progression)**
- **What supports/justifies your answer? (justify and communicate conclusions to others)**
- **Does this make mathematical sense? (making plausible arguments)**
- **Are there exceptions to your conjectures? (counterexamples, special cases)**
- **Ask questions to clarify other students work. (ask useful questions to clarify or improve the arguments)**
- **Does your answer make mathematical sense?**

DOES THIS MAKE MATHEMATICAL SENSE?

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

For the teacher:

- **Incorporate real world problems.**
- **Guide students in selecting the correct model. (graphs, tables, equations, formulas, etc.)**
- **Allow time for students to reflect and interpret their results.**

For the students:

- **What prior knowledge of the subject do you have? (mathematically proficient)**
- **Create a model of the situation using diagrams, tables, graphs, equations, formulas, etc. (Students may need a rough draft/plan of where to begin)**
- **Approximate what the outcome could be.**
- **Does your answer make mathematical sense?**
- **Do you need to revise your model to make better mathematical sense?**

DOES THIS MAKE MATHEMATICAL SENSE?

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

For the teacher:

- **Provide guidance for choosing appropriate tools.**
- **Ask appropriate questions that will allow students to apply the results they find to make additional predictions.**

For the student:

- **Which tool do you need to best solve the problem? (calculator, protractor, ruler, computer, etc)**
- **Create an equation. (mathematical model)**
- **Create a graph. (which tool would be most beneficial?)**
- **What conclusions/predictions can you make based on the results you have found from your equation and the tools you have used to solve the problem? (predictions)**
- **Does your answer make mathematical sense? / Have you used your tools correctly? (detect possible errors)**

DOES THIS MAKE MATHEMATICAL SENSE?

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

For the teacher:

- **Stress appropriate vocabulary.**
- **Stress the importance of proper notation.**
- **Stress the importance of the correct units of measure.**
- **Guide the analysis of the directions for the problems.**

For the student:

- **Use appropriate vocabulary. (communicate precisely)**
- **Support your answers. (definitions)**
- **Label graphs, diagrams, charts, figures, etc appropriately.**
- **Use appropriate units of measure.**
- **What is the meaning of the symbols ie =, + , $\cong$, $\perp$, <, >, ≤, ≥, π, ~, ±, °, ≈, ≠, $\angle$, √, $\cap$, $\cup$, $\in$, %, Δ? (state the meaning of the symbols they choose)**
- **Know the difference between exact and approximate when you are answering questions. (calculate accurately and efficiently)**
- **Does your answer make mathematical sense?**

DOES THIS MAKE MATHEMATICAL SENSE?

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

For the teacher:

- **Help students recognize patterns.**
- **Help students recognize boundaries based on the original problem – look at the big picture. (the limits, constraints based on the structure of the problems)**
- **Help the student relate back to a simpler problem. (Example: find a common denominator for higher math functions relating back to a basic math problem)**
- **Help students look at problems from various perspectives. (as a whole or as the composition of parts)**

For the student:

- **Look for a pattern in the problem.**
- **You should look for more than one way to solve the problem. (shift perspective)**
- **Would it help if I break the whole into parts or combine the parts into whole?**
- **Do I need to add anything? (a line or segment, a number to both sides, etc.)**
- **Does your answer make mathematical sense?**

DOES THIS MAKE MATHEMATICAL SENSE?

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

For the teacher:

- **Stress detail.**
- **Encourage students to find and explain justifiable shortcuts.**

For the student:

- **Pay attention to detail. (maintain oversight of the process while attending to the details)**
- **Look for patterns. (notice if calculations are repeated)**
- **Recognize and apply repeated math concepts and develop a general formula for the observation. (Notice regularity in a problem)**
- **Look for and devise shortcuts that correctly replicate the process.**
- **Does your answer make mathematical sense? (evaluate the reasonableness of their intermediate results)**

DOES THIS MAKE MATHEMATICAL SENSE?